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# Massless and Massive Quanta Resulting from a Mediumlike Metric Tensor

**Josip Šoln**[1]

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We formulate a simple model of the "primordial" scalar field theory in which the metric tensor is a generalization of the metric tensor from electrodynamics in a medium. The radiation signal corresponding to the scalar field propagates with a velocity that is generally less than $c$. This signal can be associated simultaneously with imaginary and real effective (momentum-dependent) masses. The requirement that the imaginary effective mass vanishes, which we take to be the prerequisite for the vacuumlike signal propagation, leads to the "spontaneous" splitting of the metric tensor into two distinct metric tensors: one metric tensor gives rise to masslesslike radiation and the other to a massive particle.

## 1. INTRODUCTION

There has been considerable interest in recent years in explaining the origin of the mass. The reason for this interest lies in the belief that only after we understand "where the mass comes from" might we be able to construct theories which would unify weak, electromagnetic, strong, and perhaps gravitational interactions.

In this paper we tackle the question of the origin of the mass starting from a well-known premise: mass and energy are equivalent. What we assume here is that primordially energy existed in the simplest, the radiation form, and that some primordial metric tensor evolving into a vacuumlike metric tensor spontaneously chose the primordial radiation to remain radiation or to become a massive particle. For the primordial metric tensor we choose a tensor that is a generalization of the metric tensor associated with electromagnetic radiation in a medium. (Schwinger et al., 1976; Šoln, 1978, 1981, 1982; Watson and Jauch, 1949.) For the sake of completeness, let us

[1]Harry Diamond Laboratories, Adelphi, Maryland 20783.

write down the metric tensor associated with the electromagnetic radiation in a moving medium (Schwinger et al., 1976; Šoln, 1978, 1981, 1982; Watson and Jauch, 1949).

$$\eta^{\mu\nu} = g^{\mu\nu} + (1 - n^2)u^\mu u^\nu$$
$$\mathbf{u} = \gamma \mathbf{v}, \qquad u^4 = \gamma, \qquad \gamma^2 = (1 - \mathbf{v}^2)^{-1} \tag{1}$$

Here $\mathbf{v}$ is the ordinary velocity of the medium, $g^{\mu\nu}$ is the usual metric tensor with diagonal elements $(1, 1, 1, -1)$, and $n$, the Lorentz invariant index of refraction. It is clear that in regard to the electromagnetic radiation signal (Schwinger et al., 1976; Šoln, 1978, 1981, 1982), as well as to its becoming a confined massive vector boson (Šoln, 1981, 1982), the mass of the medium is irrelevant. Hence, when we say "a medium at rest," $\mathbf{u}^\mu = (\mathbf{0}, 1)$, we formally mean a specific form for $\eta^{\mu\nu}$, $\eta^{\mu\nu} = \text{diag}\ (1, 1, 1, -n^2)$. One can easily see that regardless of what $u^\mu$ is, $n^2 = \det(\eta^\mu_\nu) \equiv \eta$. Thus with $\eta^{\mu\nu}$ as defined by (1), $n^2$ has to be a Lorentz-invariant quantity. Hence, when we conclude that a radiation signal becomes a confined (in a medium) massive particle, it is clear that this conclusion (and the numerical value for its mass) holds equally well in any frame of reference.

For a primordial medium (filled with a "dense" primordial radiation), $\eta^{\mu\nu}$ may depend on two Lorentz-invariant parameters, rather than just on one, as is the case with electromagnetic radiation. The primordial radiation signal is associated with primordial, say, scalar field $\phi$, which makes the discussion as simple as possible. One finds that scalar field $\phi$ has a dual nature: it is associated with imaginary and real, generally momentum-dependent, "masses" both propagating with the same velocity. The requirement that the (momentum-dependent) imaginary mass vanishes, which we take to be a prerequisite for the vacuumlike signal propagation, leads to a "spontaneous quantization" of the metric tensor: one metric tensor gives rise to masslesslike radiation and the other to a massive particle.

In Section 2 the specifics of the Lagrangian with primordial scalar field $\phi$ and primordial metric tensor $\eta^{\mu\nu}$ are given. The "evolution" of the primordial medium into the vacuumlike media (containing ordinarylike radiation or massive pointlike particles) is discussed in Section 3. How the structure of the primoridal medium can determine what the values of masses of confined particles are, the discussion and the conclusion are given in Section 4. In the Appendix details of the formulation of a classical field equation in auxiliary Hilbert spaces with basis vectors $|x\rangle$ and $|p\rangle$ are elaborated upon. This formulation allows us to write down simultaneously the Lagrangian in $x$ and $p$ representations, which in turn makes it "simple" to deal with "nonlocal" (in $x$ space) metric tensors, if so desired.

## 2. PRIMORDIAL MEDIUM

Let us assume that primordially we have a medium filled with energy in the form of radiation which, for simplicity, we associate with some scalar field $\phi(x)$. In analogy to a medium with electromagnetic radiation, we associate with a primordial medium the metric tensor [compare with (1)]

$$\eta^{\mu\nu}=\alpha g^{\mu\nu}+\delta u^{\mu}u^{\nu}, \tag{2}$$

where $\alpha$ and $\delta$ are, in general, two real independent and Lorentz-invariant parameters. As in (1) $u^{\mu}$ is the four-velocity of the medium with respect to some arbitrary frame of reference. We can rewrite (2) in a form similar to (1), by noticing that $\delta=\alpha(1-\eta\alpha^{-4})$, where again $\eta=\det(\eta^{\mu}_{\nu})$. Hence we write $\eta^{\mu\nu}$ and its inverse, $(\eta^{-1})^{\mu\nu}$, as (with $\alpha$ and $\beta$ again being real parameters)

$$\eta^{\mu\nu}=\alpha[g^{\mu\nu}+(1-\beta)u^{\mu}u^{\nu}] \tag{3a}$$

$$(\eta^{-1})^{\mu\nu}=\alpha^{-1}[g^{\mu\nu}+(1-\beta^{-1})u^{\mu}u^{\nu}] \tag{3b}$$

where $\beta=\eta\alpha^{-4}$, and in analogy to (1), we take that $\beta\geq 0$. We notice that $\alpha$ and $\alpha^{-1}$ are overall multiplicative factors in (3a) and (3b), respectively. (As such, we do not expect them to play fundamental roles in the evolution of the primordial medium.)

It is clear that interchange $\beta\leftrightarrow\beta^{-1}$ (which implies interchanges $\alpha\leftrightarrow\alpha^{-1}$ and $\eta\leftrightarrow\eta^{-1}$) induces interchange $\eta^{\mu\nu}\leftrightarrow(\eta^{-1})^{\mu\nu}$. This suggests that we may restrict $\beta$ as $0\leq\beta\leq 1$. Namely, when $1\leq\beta\leq\infty$, then $0\leq\beta^{-1}\leq 1$, and we can define $\beta'=\beta^{-1}$, $\alpha'=\alpha^{-1}$ giving $\eta'^{\mu\nu}=(\eta^{-1})^{\mu\nu}$, which reduces the case of $1\leq\beta\leq\infty$ back to the case of $0\leq\beta\leq 1$. Thus, with restriction $0\leq\beta\leq 1$, we expect that both $\eta^{\mu\nu}$ and $(\eta^{-1})^{\mu\nu}$ should be used in defining equations of motion for a primordial scalar field. Now in order that $\eta^{\mu\nu}$ and $(\eta^{-1})^{\mu\nu}$ exist, we require that a real $\alpha$ also satisfies $\alpha\neq 0, \pm\infty$. Then because $\beta\geq 0$, we also have that $\eta=\beta\alpha^{4}\geq 0$. However, we can still have a problem with $(\eta^{-1})^{\mu\nu}$ when $\beta\to 0$. Hence, the range of $\beta$, $0\leq\beta\leq 1$ will be understood as $\varepsilon\leq\beta\leq 1$, with $\varepsilon$ small and positive. The limit $\varepsilon\to +0$ will be understood when $\beta=0$ is specified.

Hence, viewing classical fields as matrix elements of abstract field operators (see the Appendix for details), we couple abstract field operators $\phi(\hat{x})$ and $\psi(\hat{y})$ to $(\eta^{-1})^{\mu\nu}$ and $\eta^{\mu\nu}$, respectively. The corresponding abstract Lagrangian density operators are taken to be

$$L(\hat{x})=-\tfrac{1}{2}\phi(\hat{x})\hat{p}_{\mu}(\eta^{-1})^{\mu\nu}\hat{p}_{\nu}\phi(\hat{x}), \tag{4a}$$

$$L'(\hat{y})=-\tfrac{1}{2}\psi(\hat{y})\hat{P}_{\mu}\eta^{\mu\nu}\hat{P}_{\nu}\psi(\hat{y}) \tag{4b}$$

Here we assume that the specific numerical values of the two Lorentz-invariant parameters $\alpha$ and $\beta$ (or $\alpha$ and $\eta$) will define the physical content of the theory. As described in the Appendix, we have that

$$[\hat{x}^\mu, \hat{p}^\nu] = ig^{\mu\nu} \tag{5a}$$

$$[\hat{y}^\mu, \hat{P}^\nu] = ig^{\mu\nu} \tag{5b}$$

Clearly from (4a) and (4b) we see that $\phi(x)$ and $\psi(y)$ ($x$ and $y$ being respective four-coordinates) are primordial fields that correspond to $0 \leq \beta \leq 1$ and to $1 \leq \beta' \leq \infty$, where $\beta' = \beta^{-1}$. Although fields $\phi(x)$ and $\psi(y)$ correspond to $\beta$ and $\beta'$ whose ranges are different, because we defined $\beta' = \beta^{-1}$, $\phi(\hat{x})$ and $\psi(\hat{y})$ can be related. Namely, $L'(\hat{y})$ from (4b) can be written as (for notations, see the Appendix)

$$L'(\hat{y}) = -\tfrac{1}{2}\psi(\hat{y})(\hat{P} \cdot \eta) \cdot \eta^{-1} \cdot (\eta \cdot \hat{P})\psi(\hat{y})$$

which upon comparison with (4a) gives

$$\phi(\hat{x}) = \psi(\hat{y}) \tag{6}$$

$$L(\hat{x}) = L'(\hat{y}) \tag{7}$$

and

$$\hat{p}^\mu = \eta^\mu_\nu \hat{P}^\nu \tag{8}$$

However, in order that (8) be consistent with (5a) and (5b), we must have

$$\hat{y} = \eta^\mu_\nu \hat{x}^\nu \tag{9}$$

In view of (8) and (9), it is clear that the corresponding basis vectors are related as (see the Appendix)

$$|x\rangle = \eta^{1/2}|y\rangle \tag{10a}$$

$$|P\rangle = \eta^{1/2}|p\rangle \tag{10b}$$

In order to arrive at equations of motion in configuration and momentum spaces, as indicated in the Appendix, we write down the relevant actions as

$$A = (2\pi)^4\langle p = 0|L(\hat{x})|p = 0\rangle$$
$$= \int d^4x \mathscr{L}(x) = \frac{1}{(2\pi)^4}\int d^4p\, \mathscr{L}(p) \tag{11a}$$

$$A' = (2\pi)^4\langle P = 0|L'(\hat{y})|P = 0\rangle$$
$$= \int d^4y\, \mathscr{L}'(y) = \frac{1}{(2\pi)^4}\int d^4P \mathscr{L}'(P) \tag{11b}$$

From (11a) and (11b) we conclude that

$$\mathscr{L}(x) = -\tfrac{1}{2}[\partial_\mu\phi(x)](\eta^{-1})^{\mu\nu}[\partial_\nu\phi(x)] \tag{12a}$$

$$\mathscr{L}'(y) = -\tfrac{1}{2}[\partial_\mu\psi(y)]\eta^{\mu\nu}(\partial_\nu\psi(y)] \tag{12b}$$

and

$$\mathscr{L}(p) = -\tfrac{1}{2}\phi(-p)p_\mu(\eta^{-1})^{\mu\nu}p_\nu\phi(p) \tag{13a}$$

$$\mathscr{L}'(P) = -\tfrac{1}{2}\psi(-P)P_\mu\eta^{\mu\nu}P_\nu\psi(P) \tag{13b}$$

It is easy to see that

$$A' = \eta A \tag{14}$$

and

$$\psi(P) = \eta\phi(\eta \cdot P) \tag{15}$$

Restricting ourselves to the equations of motion in momentum space, one easily arrives at

$$p_\mu(\eta^{-1})^{\mu\nu}p_\nu\phi(p) = 0 \tag{16a}$$

$$P_\mu\eta^{\mu\nu}P_\nu\psi(P) = 0 \tag{16b}$$

which are satisfied with [cf. (15)]

$$\phi(p) = a(p)\delta\big(p_\mu(\eta^{-1})^{\mu\nu}p_\nu\big) \tag{17a}$$

$$\psi(P) = \eta a(\eta \cdot P)\delta(P_\mu\eta^{\mu\nu}P_\nu) \tag{17b}$$

Here $a(p)$ is some Lorentz-invariant scalar.

Equations (17a) and (17b) imply respective "mass-shell" conditions, which we write as

$$p_\mu(\eta^{-1})^{\mu\nu}p_\nu = \alpha^{-1}[p^2 + m^2(\beta)] = 0 \tag{18a}$$

$$m^2(\beta) = (1-\beta^{-1})(p\cdot u)^2 = (1-\beta^{-1})\omega_0^2 \tag{19a}$$

$$P_\mu\eta^{\mu\nu}P_\nu = \alpha[P^2 + M^2(\beta)] = 0 \tag{18b}$$

$$M^2(\beta) = (1-\beta)(P\cdot u)^2 = (1-\beta)\Omega_0^2 \tag{19b}$$

where $\omega_0(\Omega_0)$ is $p^4(P^4)$ in the "rest frame" of the primordial medium, $\omega_0 = -(p\cdot u)$ $[\Omega_0 = -(P\cdot u)]$. The quantities $m(\beta)$ and $M(\beta)$ can be considered as effective, momentum-dependent masses. Because of $0 \le \beta \le 1$, they obviously satisfy

$$m^2(\beta) \le 0 \tag{20a}$$

$$M^2(\beta) \ge 0 \tag{20b}$$

In fact, because $p^\mu = (\eta\cdot P)^\mu$, we have

$$\begin{aligned} m^2(\beta) &= -\alpha^2\beta M^2(\beta) \\ &= -\alpha^2\beta(1-\beta)\Omega_0^2 \end{aligned} \tag{21}$$

implying

$$\alpha^2 > 0 \tag{22}$$

Hence, only real $\alpha$'s are admitted. As we see, because $0 \leq \beta \leq 1$, $m(\beta)$ is imaginary or zero, while $M(\beta)$ is real or zero.

At this point we notice that the effective masses $m(\beta)$ and $M(\beta)$ are given directly in terms of Lorentz-invariant $\omega_0 = -(p \cdot u)$ and $\Omega_0 = -(P \cdot u)$, respectively [equations (19a) and (19b)]. In order to allow for more "complicated" structures of the primordial medium, we shall take that $\omega_0$ and $\Omega_0$ themselves may depend on Lorentz-invariant $\alpha$ and $\beta$ (or $\eta$). However, since $\alpha$ is just an overall multiplicative factor in the definition of $\eta^{\mu\nu}$, equation (3a), we expect that only the dependence on $\beta$ (or $\eta$) is important. In what follows, these specific dependences of $\omega_0$ and $\Omega_0$ on $\beta(\eta)$ and possibly on $\alpha$ will not be denoted explicitly, except when necessary for clarity.

Let us for a moment restrict our discussion to the rest frame of the primordial medium: $u^\mu = (\mathbf{0}, 1)$. Here, $\eta^{\mu\nu} = \alpha\,\mathrm{diag}(1, 1, 1, -\beta)$ and $(\eta^{-1})^{\mu\nu} = \alpha^{-1}\,\mathrm{diag}(1, 1, 1, -\beta^{-1})$. Now the momenta can be written as

$$p_0^\mu = (\mathbf{p}_0, \omega_0), \qquad P_0^\mu = (\mathbf{P}_0, \Omega_0)$$

and mass-shell conditions (18a) and (18b) become

$$\alpha^{-1}[\mathbf{p}_0^2 - \beta^{-1}\omega_0^2] = 0 \tag{23a}$$

$$\alpha[\mathbf{P}_0^2 - \beta\Omega_0^2] = 0 \tag{23b}$$

respectively. Because $\alpha \neq 0, \pm\infty$, we solve (23a) and (23b) as

$$\mathbf{p}_0\beta^{1/2} = \mathbf{s}_0\omega_0 \tag{24a}$$

$$\mathbf{P}_0 = \mathbf{s}_0\Omega_0\beta^{1/2} \tag{24b}$$

where $\mathbf{s}_0^2 = 1$. These solutions are consistent with $p_0^\mu = (\eta \cdot P_0)^\mu$:

$$\mathbf{p}_0 = \alpha\mathbf{P}_0, \qquad \omega_0 = \alpha\beta\Omega_0 \tag{25}$$

Now because $m(\beta)$ is imaginary, we shall define the velocity of the primordial radiation signal associated with $\phi(x; p_0) \sim \exp(ix \cdot p_0)$ as a group velocity

$$\mathbf{w}_0 = \frac{\partial\omega_0}{\partial\mathbf{p}_0} = \mathbf{s}_0\beta^{1/2}, \qquad w_0^4 = 1$$
$$\omega_0 = \mathbf{w}_0 \cdot \mathbf{p}_0 \tag{26}$$

On the other hand, comparison of (26) with (24b) immediately gives

$$\mathbf{w}_0 = \mathbf{s}_0\beta^{1/2} = \frac{\mathbf{P}_0}{\Omega_0} \tag{27}$$

Relation (27) states that when the effective momentum-dependent mass is real, $M^2(\beta) \geq 0$, the velocity of the corresponding primordial radiation signal, associated here with $\psi(y; P_0) \sim \exp(iy \cdot P_0)$, is a particlelike velocity. In view of $0 \le \beta \le 1$, we have the important relation

$$|\mathbf{w}_0| = \beta^{1/2} \le 1 \tag{28}$$

Now, since under $\beta \leftrightarrow \beta^{-1}$, $m^2(\beta)$ and $M^2(\beta)$ get interchanged, we then conclude that when $1 \le \beta \le \infty$, we have

$$|\mathbf{w}_0| = \beta^{-1/2} \le 1 \tag{29}$$

As we see, the primordial field theory as presented here actually limits the velocity of the signal to less than or equal to $c$.

However, we know that the momentum-dependent masses $m(\beta)$ and $M(\beta)$ are Lorentz-invariant quantities. Hence we expect that the definition of velocity of the primordial radiation signal should be independent of the frame of reference. Indeed, by virtue of (18a) or (18b), we have that

$$p \cdot P = 0 \Rightarrow \omega = \frac{\mathbf{P}}{\Omega} \cdot \mathbf{p} \tag{30}$$

where in the arbitrary frame of reference we have written the momenta as

$$p^\mu = (\mathbf{p}, p^4 = \omega), \qquad P^\mu = (\mathbf{P}, P^4 = \Omega)$$

Hence, with the same reasoning as before, we conclude that the velocity of the primordial radiation signal in an arbitrary frame is (notice, $0 \le \beta \le 1$)

$$\mathbf{w} = \frac{\partial \omega}{\partial \mathbf{p}} = \frac{\mathbf{P}}{\Omega}, \qquad w^4 = 1 \tag{31}$$

satisfying

$$w_\mu \eta^{\mu\nu} w_\nu = 0 \tag{32}$$

Again we see that $\mathbf{w}$ is simultaneously a grouplike velocity (in momentum $p^\mu$) and a particlelike velocity (in momentum $P^\mu$). Furthermore, from equation (32) we see that indeed $\eta^{\mu\nu}$ is a metric tensor of a primordial medium filled with primordial radiation. [When $1 \le \beta \le \infty$, then understandably $(\eta^{-1})^{\mu\nu}$ is a metric tensor which can be transformed back into $\eta^{\mu\nu}$ with $\beta \leftrightarrow \beta^{-1}$.]

Now we would like to find how $\mathbf{w}_0$ and $\mathbf{w}$ are related. This is most easily found by means of Lorentz transformations which take us from the rest frame of the primordial medium to an arbitrary frame—a frame with respect to which the primordial medium is moving with four velocity $u^\mu$.

The necessary transformation matrices are

$$\Lambda^{i\mu} = g^{i\mu} + \frac{u^i}{1+\gamma}(u^\mu - g^{4\mu}), \qquad \Lambda^{4\mu} = -u^\mu \tag{33a}$$

$$\hat{\Lambda}^{\nu\mu} = \Lambda^{\nu\mu}(u \to \tilde{u}), \qquad \tilde{u}^\mu = \gamma(-\mathbf{v}, 1) \tag{33b}$$

$$g_{\mu\nu}\Lambda^{\rho\mu}\Lambda^{\sigma\nu} = g_{\mu\nu}\tilde{\Lambda}^{\rho\mu}\tilde{\Lambda}^{\sigma\nu} = g^{\rho\sigma} \tag{34}$$

$$g_{\rho\sigma}\Lambda^{\rho\mu}\Lambda^{\sigma\nu} = g_{\rho\sigma}\tilde{\Lambda}^{\rho\mu}\tilde{\Lambda}^{\sigma\nu} = g^{\mu\nu} \tag{35}$$

Momenta $P^\mu$ and $P_0^\mu$ are related as

$$P^\mu = \tilde{\Lambda}^{\mu\nu}P_\nu^0, \qquad P_0^\mu = \Lambda^{\mu\nu}P_\nu$$

giving specifically [cf. (24b)]

$$P^i = \Omega_0\left\{\gamma v^i + \beta^{1/2}\left[s_0^i + \frac{v_i(\gamma-1)(\mathbf{s}_0\cdot\mathbf{v})}{\mathrm{v}^2}\right]\right\}$$

$$\Omega = \gamma\Omega_0[1+\beta^{1/2}(\mathbf{s}_0\cdot\mathbf{v})] \tag{36}$$

By virtue of (31) we have

$$\mathbf{w} = \frac{\gamma v^2\mathbf{v} + \beta^{1/2}[v^2\mathbf{s}_0 + \mathbf{v}(\gamma-1)(\mathbf{s}_0\cdot\mathbf{v})]}{\gamma v^2[1+\beta^{1/2}(\mathbf{s}_0\cdot\mathbf{v})]}$$

$$w^4 = 1 \tag{37}$$

Because $P\cdot\eta\cdot P$ is a Lorentz-invariant quantity, $P\cdot\eta\cdot P = P_0\cdot\eta_0\cdot P_0 = 0$, so $w^\mu$ from (37) satisfies (32) explicitly. Also we have that $|\mathbf{w}| \leq 1$. Velocity $\mathbf{w}$ depends on the direction of $\mathbf{P}$. If we fix $\mathbf{P}$, then $\eta a(\eta\cdot P)$ from (17b) is understood to be sharply peaked ($\delta$-like function) around this $\mathbf{P}$. One should also notice that while $\mathbf{w}$ depends explicitly on $\beta$, it does not depend on $\alpha$.

## 3. EVOLUTION OF THE PRIMORDIAL MEDIUM INTO THE VACUUMLIKE MEDIA

We define a classical vacuum as a medium in which no imaginary masses (dependent or independent of momenta) occur. Since for $0 \leq \beta \leq 1$, $m^2(\beta) \leq 0$, this means that we are looking for those $\beta$'s for which $m^2(\beta) = 0$. Furthermore, on physical grounds we assume that $\Omega_0$ is finite for $0 \leq \beta \leq 1$ (in particular, $\Omega_0 \neq \infty$ for $\beta = 0$ and 1). Then by virtue of equation (21), $m^2(\beta) = 0$ for $\beta = 1$ or 0. This in turn leads to "spontaneous quantization" of $\eta^{\mu\nu}$ into two distinct metric tensors:

$$\eta^{\mu\nu}(\beta = 1) = \alpha g^{\mu\nu} \tag{38}$$

$$\eta^{\mu\nu}(\beta = 0) = \alpha g_+^{\mu\nu}, \qquad g_+^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \tag{39}$$

So far the only constraint on $\alpha$ is that it has to be real and different from 0 and $\pm\infty$ [cf. (3a) and (3b)]. Hence, for either of cases (38) and (39) it would appear that the choice of $|\alpha|=1$ makes the whole discussion as general as it can be. In regard to (38) and (39), we shall decide on specific values for $\alpha$ by carrying out the quantization of corresponding fields. Next we analyze cases (38) and (39) in somewhat more detail.

Setting $\beta=1$ in (8), (13a), (13b), and (15), we obtain

$$\mathcal{L}(p;\beta=1)=-\frac{1}{2\alpha}\phi(-p)p_\mu g^{\mu\nu}p_\nu\phi(p) \tag{40a}$$

$$\mathcal{L}'(P;\beta=1)=-\frac{\alpha}{2}\psi(-P)P_\mu g^{\mu\nu}P_\nu\psi(P) \tag{40b}$$

$$p^\mu=\alpha P^\mu \tag{41a}$$

$$y^\mu=\alpha x^\mu \tag{41b}$$

$$\psi(P)=\alpha^4\phi(p) \tag{41c}$$

As we see, the case of $\beta=1$ clearly corresponds to a massless radiation since $m=M=0$ [cf. (19a) and (19b)]. Furthermore from (37) we obtain

$$\mathbf{w}(\beta=1)=\frac{\mathbf{s}_0 v^2+\mathbf{v}[\gamma v^2+(\gamma-1)(\mathbf{s}_0\cdot\mathbf{v})]}{\gamma v^2(1+\mathbf{s}_0\cdot\mathbf{v})} \tag{42a}$$

$$|\mathbf{w}(\beta=1)|=1 \tag{42b}$$

where (42b) holds regardless of what the directions of $\mathbf{v}$ and $\mathbf{s}_0$ are. Hence, with $\beta=1$ the radiation signal clearly propagates with the velocity of light regardless of what the reference frame is and regardless of whether Lagrangian (40a) or (40b) is used for its description. In fact, Lagrangians (40a) and (40b) are basically the same except that momenta which enter into them are rescaled with respect to each other. (The theory is invariant under scale transformations since $m=M=0$.)

Let us now decide on the value for $\alpha$ in (40a) and (40b) by quantizing $\phi$ and $\psi$. Consistent with the canonical formalism (Nagy, 1966), we obtain, separately for $\phi$ and $\psi$, in momentum spaces (Bogoliubov and Shirkov, 1980; Šoln, 1983)

$$[\phi(p),\phi(p')]=i\alpha(2\pi)^4\delta_{(4)}(p+p')D(p) \tag{43a}$$

$$[\psi(P),\psi(P')]=\frac{i}{\alpha}(2\pi)^4\delta_{(4)}(P+P')D(P) \tag{43b}$$

$$\begin{aligned}D(k)&=-i4\pi\varepsilon(k^4)\delta(k^2),\\ \varepsilon(k^4)&=\tfrac{1}{2}[\theta(k^4)-\theta(-k^4)]\end{aligned} \tag{44}$$

When applying (41a) and (41c) to (43a), we must obtain (43b). By taking into account that $\varepsilon(\alpha k^4) = (\alpha/|\alpha|)\varepsilon(k^4)$, we see that this will happen if $|\alpha|^3\alpha = 1$. The only real $\alpha$ that satisfies this is $\alpha = 1$. Hence, the resulting field theory will be described with the positive metric in Hilbert space.

Let us now turn our attention to the case when $\beta = 0$. Now $\eta^{\mu\nu} = \alpha g_+^{\mu\nu}$ [equation (39)] where $g_+^{\mu\nu}$ satisfies $g^{\mu}_{+\rho} g_+^{\rho\nu} = g_+^{\mu\nu}$, $(g_+ \cdot u)^\mu = 0$. Thus $u \cdot p = \alpha u \cdot g_+ \cdot P = -\omega_0 = 0$. Since $m(\beta = 0) = 0$, we also have $p^2 = p_0^2 = 0$ [cf. (18a)], which then also implies $p_0^i = 0$. Now applying Lorentz transformations [equation (33b)] to $p_0^\mu = 0$, we obtain that in any frame $p^\mu = 0 = \alpha(g_+ \cdot P)^\mu = \alpha[P^\mu + u^\mu(u \cdot P)]$. Hence for $\beta = 0$ we have

$$p^\mu = 0 \tag{45}$$

and

$$P^\mu = u^\mu \Omega_0(\beta \to 0) \tag{46a}$$

$$M(\beta = 0) = \Omega_0(\beta \to 0) \tag{46b}$$

where $\Omega_0(\beta \to 0) = \lim \Omega_0(\beta)$, as $\beta \to 0$ if $\Omega_0$ depends on $\beta$. By setting $\beta = 0$ in (36), one verifies (46a) directly, while from (37) one obtains

$$\mathbf{w}(\beta = 0) = \mathbf{v} \tag{47}$$

It is evident that the primordial radiation signal becomes confined in the primordial medium for which $\beta = 0$. This signal behaves like a massive particle whose mass is simply the angular frequency, $\Omega_0$, in the rest frame of the primordial medium.

Next we investigate what is the trajectory that this confined massive particle follows. This particle is located at space-time point $x^\mu$ which is formally complementary to $p^\mu = 0$ [cf. (4a)]. Another space-time point, $y^\mu$, which is formally complementary to $P^\mu$ [cf. (4b)], is given as $y^\mu = \alpha(g_+ \cdot x)^\mu$ when $\beta = 0$. With $x^4 = t_x$, $y^4 = t_y$, etc., we can write

$$t_x = \alpha^{-1} t_y + \gamma\tau \tag{48a}$$

$$\mathbf{x} = \alpha^{-1}\mathbf{y} + \gamma\mathbf{v}\tau \tag{48b}$$

where $t_y^0 = -(y \cdot u) = 0$ and $t_x^0 = \tau = -(x \cdot u)$, with $\tau$ being the proper time. One easily sees that (48a) is consistent with a Lorentz transformation relating $\tau$ and $t_x$. Namely, because $t_y^0 = 0$, with (33a) and (33b) we see that $t_y = \mathbf{v} \cdot \mathbf{y}$ and $\mathbf{y} = \gamma\mathbf{y}_0$. Here the position of the particle in the rest frame of the medium with $\beta = 0$ is $\mathbf{x}_0 = \alpha^{-1}\mathbf{y}_0$. Combining these facts with (48b), we obtain

$$t_y = \alpha(\mathbf{x} \cdot \mathbf{v} - \gamma v^2 \tau) \tag{49}$$

which, when combined with (48a), yields

$$\tau = \gamma(t_x - \mathbf{v} \cdot \mathbf{x}) \tag{48a$'$}$$

Finally we write (48a) and (48b) as

$$t_x = \gamma(\alpha^{-1}\mathbf{v} \cdot \mathbf{y}_0 + \tau) \tag{50a}$$

$$\mathbf{x} = \gamma(\alpha^{-1}\mathbf{y}_0 + \mathbf{v}\tau) \tag{50b}$$

Equation (50b) is a statement about the evolution of the position of the particle in an arbitrary frame, where the particle was located at $\mathbf{x}_0 = \alpha^{-1}\mathbf{y}_0$ in the rest frame of the medium. (One notices that if $\mathbf{y}_0 = 0$ is chosen then $\mathbf{y} = 0$, $t_y = 0$, $t_x = \gamma\tau$, and $\mathbf{x} = \gamma\mathbf{v}\tau$.) Finally, since $\alpha^{-1}$ is essentially rescaling $\mathbf{y}_0$ we could put $\alpha = \pm 1$.

Despite the fact that $\eta^{\mu\nu}$ becomes essentially a projection tensor, we can still describe a confined massive particle in a $\beta = 0$ medium with a Lagrangian. To do so we notice that getting $u^\mu$ off the "mass shell," $u^2 \neq -1$, means getting $P^\mu$ off the mass shell, $P^2 \neq -M^2$. Now

$$P \cdot \eta(\beta = 0) \cdot P = \alpha(P^2 + M^2)(P^2/M^2)$$

Clearly,

$$P \cdot \eta(\beta = 0) \cdot P/(P^2 + M^2) \to -\alpha, \qquad \text{as } P^2 \to -M^2$$

Therefore for $\mathscr{L}'(P; \beta = 0)$ we can simply write

$$\mathscr{L}'(P; \beta = 0) = \frac{\alpha}{2}\psi(-P)(P^2 + M^2)\psi(P) \tag{51}$$

In analogy to the $\beta = 1$ case, the quantization of $\psi(P)$ now requires (Nagy, 1966; Bogoliubov and Shirkov, 1980; Šoln, 1983)

$$\begin{aligned} [\psi(P), \psi(P')] &= -\frac{i}{\alpha}(2\Delta)^4\delta_{(4)}(P + P')\Delta(P; M^2) \\ \Delta(P, M^2) &= -i4\pi\varepsilon(P^4)\delta(P^2 + M^2) \end{aligned} \tag{52}$$

Hence, if consistent with the $\beta = 1$ case the value of $\alpha$ is fixed as $\alpha = 1$, the resulting theory for the $\beta = 0$ case (massive particle) will be described with the negative metric in the Hilbert space.

We know that most quantum theories with massive particles are described with positive metrics in their respective Hilbert spaces. Now we would like to know whether, at least in the mathematical sense, it is possible to find $\alpha$ as a function of $\eta$ which allows us to have positive metrics in the Hilbert space when $\beta \to 0, +\infty$. Keeping in mind that originally $\beta$ and $\eta$ were allowed to vary between zero and plus infinity, we write for $\alpha$ the following expression:

$$\begin{aligned} \alpha(\eta, \eta_0) &= \theta(1 - \eta)\frac{\eta - \eta_0}{|\eta - \eta_0|} + \theta(\eta - 1)\frac{\eta_0^{-1} - \eta}{|\eta - \eta_0^{-1}|} \\ &0 < \eta_0 < 1, \qquad \eta \geq 0 \end{aligned} \tag{53}$$

Taking into account that $\theta(1-\eta^{-1})=\theta(\eta-1)$ and $\theta(\eta^{-1}-1)=\theta(1-\eta)$, we see that

$$\begin{aligned}\alpha(\eta,\eta_0)&=\alpha(\eta^{-1},\eta_0)=\alpha^{-1}(\eta,\eta_0)\\ \alpha^2(\eta,\eta_0)&=1\end{aligned}\tag{54}$$

Hence, because of (54), $\beta=\eta$ [compare with the text after relation (3b)]. One easily verifies that

$$\varepsilon\le\eta\le 1-\varepsilon,\ \varepsilon\to+0:\ \alpha(\eta,\eta_0)=\begin{cases}1, & \eta>\eta_0\\ -1, & \eta<\eta_0\end{cases}\tag{55a}$$

$$1+\varepsilon\le\eta\le+\infty,\ \varepsilon\to+0:\ \alpha(\eta,\eta_0)=\begin{cases}1, & \eta<\eta_0^{-1}\\ -1, & \eta>\eta_0^{-1}\end{cases}\tag{55b}$$

When a primordial medium with some $\eta$ evolves into the usual vacuumlike state, then $\eta$ will spontaneously choose 0, 1, or $+\infty$. Now as long as the evolution of $\alpha$ is governed by (53), then according to (55a) and (55b) the limiting $\alpha$ will be such that the resulting theory can be quantized with a positive metric.

An interesting situation develops if $\alpha$ is simply fixed at $-1$ throughout the evolution of the medium. Then case $\eta=1$ ($\beta=1$), which requires $\alpha=1$ for its quantization, would be excluded from the values that $\eta$ can assume spontaneously; only $\eta\to 0$ and $+\infty$ would be allowed. In this case all of the energy in the medium would end up in masses of confined particles.

## 4. STRUCTURE OF THE MEDIUM, DISCUSSION, AND CONCLUSION

What we mean by the "structure" of the primordial medium is how it behaves as a function of $\beta$. The basic structure, of course, is given by tensor $\eta^{\mu\nu}$, while the additional structure is given by the dependence of $\Omega_0$ (or $\omega_0$) on $\beta$.

Working within the usual restriction on $\beta$, $0\le\beta\le 1$, we then say that the primordial medium is of the basic, or simplest, structure if $\Omega_0$ is independent of $\beta$. Now $M^2(\beta=1)=m^2(\beta=1)=0$, and $M^2(\beta=0)=\Omega_0^2$ and $m^2(\beta=0)=0$, as expected. Hence, any $\Omega_0$ becomes the mass of the confined primordial radiation signal when $\beta\to 0$.

A very interesting structure of the primordial medium is obtained if we choose

$$\Omega_0^2(\beta)=\frac{\Omega_0^2(0)}{1-\beta},\qquad 0\le\beta<1\tag{56a}$$

$$\Omega_0^2(\beta)=\Omega_0^2(1),\qquad \beta=1\tag{56b}$$

Here, consistent with the discussion preceding equations (38) and (39) we have chosen for $\Omega_0$ at $\beta=1$ some finite value $\Omega_0(1)$. The corresponding values for $M^2(\beta)$ are then

$$M^2(\beta)=\Omega_0^2(0), \qquad 0\le\beta<1 \tag{57a}$$

$$M^2(\beta)=0, \qquad \beta=1 \tag{57b}$$

As we see, the structure of the primordial medium now is such that $M(\beta)$ is independent of $\beta$ [and $\Omega_0(\beta)$] except at $\beta\to 1$ when it goes sharply to zero. As a consequence, one could then conclude that for any $\beta(0\le\beta<1)$ the radiation signal is confined whose mass is $M=\Omega_0(0)$. However we have to remember that for a truly confined signal $m(\beta)$ has to vanish. Since $m^2(\beta)=-\alpha^2\beta\Omega_0^2(0)$, we see that this will happen only for $\beta=0$. Hence, only $\beta=0$ is the medium with a confined pointlike particle whose mass is $\Omega_0(0)$.

The fact that $M(\beta)$ is independent of $\beta$ when $0\le\beta<1$, makes the primordial medium very much like a medium of the collisionless isotropic electron plasma where $\Omega_0(0)$ becomes a plasma frequency. To see this let us look at an electromagnetic wave moving through such a plasma which has a quality of an ideal charged electron fluid with constant density in the configuration space. The four-vector electric current densities in the configuration and momentum spaces are, respectively,

$$j^\mu(y)=\frac{eN}{m_e}p_e^\mu(y) \tag{58a}$$

$$j^\mu(P)=\frac{eN}{m_e}P_e^\mu(P) \tag{58b}$$

where $m_e$ is the electron mass, $N$ the number of electrons in $\text{cm}^3$, and configuration and momentum space four-momenta of the electron are given, respectively, as

$$p_e^\mu(y)=m_e u^\mu \tag{59a}$$

$$p_e^\mu(P)=m_e u^\mu(2\pi)^4\delta_{(4)}(P) \tag{59b}$$

Relaxing the mass-shell condition for $P^\mu$, equation (18b), we write now equation (16b) for electromagnetic potential $A^\mu(P)$, however, modified with the source current term (Šoln, 1982) as follows:

$$P_\rho\eta^{\rho\sigma}P_\sigma A^\mu(P)=j^\mu(P) \tag{60}$$

where $j^\mu(P)$ is given by (58b). In equation (60), $\eta^{\mu\nu}$ has $\beta=n^2$, with $n$ being the index of refraction, $0\le n<1$, while $\alpha$ is simply absorbed into $j^\mu(P)$ (which is equivalent to setting $\alpha=1$). We see that $P\cdot\eta\cdot P\neq 0$ only for $P^{\mu}=0$, while $P\cdot\eta\cdot P=0$ otherwise. Clearly, $(u\cdot P)=-\Omega_0(\beta)$ is still a Lorentz-invariant quantity and choosing for it form (56a), which makes $M(\beta)$ independent of $\beta$, we obtain

$$[P^2+\Omega_0^2(0)]A^{\mu}(P)=\frac{eN}{m_e}p_e^{\mu}(P) \tag{61}$$

Forcing now equation (61) to be written in the "minimal gauge" form

$$P^2A^{\mu}(P)=\frac{eN}{m_e}[p_e^{\mu}(P)-eA^{\mu}(P)] \tag{62}$$

we obtain

$$\Omega_0^2(0)=\frac{e^2N}{m_e}$$

which is the plasma frequency squared. As we see, the source current in equation (60), which is different from zero only for $P^{\mu}=0$, is a very useful tool for numerically evaluating $\Omega_0(0)$. Otherwise, the radiation signal behaves like a "freelike" signal (for $P^{\mu}\neq 0$) whose confinement into a massive pointlike particle occurs only for $\beta=0$ (Šoln, 1981, 1982).

One of the more interesting points of this primordial field theory model is the fact that the velocity of primordial signal can never exceed the velocity of light. The significant point, however, is that the primordial medium in which the energy appeared in the radiationlike form is allowed to evolve into a vacuumlike media in which the energy appears in the pure radiation form ($\beta=1$) and the massive form ($\beta=0$). We believe that in high-energy collisions qualitatively one is creating within the interaction region many primordial-like media which after some time all become vacuumlike media with pure radiation or massive particles. These primordial media carry many attributes, such as charge, spin, isospin, and the like, which themselves may have a say as to what kind of $\Omega_0$ will be confined into a mass.

## APPENDIX

If one wishes to be able to describe classical fields simultaneously in configuration and momentum spaces, then the notions of auxiliary Hilbert spaces with basis vectors $|x\rangle$ and $|p\rangle$ are very useful. These basis vectors satisfy

$$\langle x|x'\rangle=\delta_{(4)}(x-x') \tag{A1}$$

$$\langle p|p'\rangle=\delta_{(4)}(p-p') \tag{A2}$$

$$\langle x|p\rangle=\frac{1}{(2\pi)^2}\exp(ix\cdot p) \tag{A3}$$

and

$$\int d^4x|x\rangle\langle x| = \int d^4p|p\rangle\langle p| = 1 \tag{A4, 5}$$

where (A4) and (A5) are the completeness relations. The position and momentum operators, $\hat{x}^\mu$ and $\hat{p}^\mu$, satisfy the following commutator:

$$[\hat{x}^\mu, \hat{p}^\nu] = ig^{\mu\nu} \tag{A6}$$

where $g^{\mu\nu} = \text{diag}(1, 1, 1, -1)$. Some further useful relations consistent with (A1)-(A6) are

$$\langle x|\hat{x}|x'\rangle = x^\mu \delta_{(4)}(x - x') \tag{A7}$$

$$\langle x|\hat{p}^\mu|x'\rangle = p^\mu(x - x') = \frac{1}{i}\frac{\partial}{\partial x_\mu}\delta_{(4)}(x - x') \tag{A8}$$

$$\langle p|\hat{x}^\mu|p'\rangle = \frac{1}{(2\pi)^4}x^\mu(p - p') = \frac{i}{(2\pi)^4}\frac{\partial}{\partial p_\mu}\delta_{(4)}(p - p') \tag{A9}$$

$$\langle p|\hat{p}^\mu|p'\rangle = p^\mu\delta_{(4)}(p - p') \tag{A10}$$

Now, to a classical field we can associate abstract field operator $\phi(\hat{x})$ with matrix elements [cf. (A9)]:

$$\langle x|\phi(\hat{x})|x'\rangle = \phi(x)\delta_{(4)}(x - x') \tag{A11}$$

$$\langle p|\phi(\hat{x})|p'\rangle = \frac{1}{(2\pi)^4}\phi(p - p') \tag{A12}$$

With (A3)-(A5) one easily verifies that

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^4}\int d^4p\, \phi(p; x) \\ \phi(p; x) &= \phi(p)\exp(ip \cdot x) \end{aligned} \tag{A13}$$

To a Lagrangian we now associate the Lagrangian abstract operator which may explicitly depend on $\hat{p}^\mu$ while its dependence on $\hat{x}$ comes only through $\phi(\hat{x})$:

$$\hat{L} = L[\phi(\hat{x}); \hat{p}] \tag{A14}$$

The action is now given as

$$\begin{aligned} A &= (2\pi)^4\langle 0|\hat{L}|0\rangle \\ &= \int d^4x\, \mathscr{L}(x) = \frac{1}{(2\pi)^4}\int d^4p\, \mathscr{L}(p) \\ |0\rangle &\equiv |p = 0\rangle \end{aligned} \tag{A15}$$

where $\mathscr{L}(x)$ and $\mathscr{L}(p)$ are Lagrangians in $x$ and $p$ spaces, respectively. The equations of motion are derived by imposing that the variation of the action, $\delta A$, be zero when a small variation of the abstract field operator, $\delta\hat{\phi}$, was made:

$$\begin{aligned} \delta A &= (2\pi)^4 \langle 0|\delta\hat{L}|0\rangle = 0 \\ \delta\hat{L} &= L(\hat{\phi}+\delta\hat{\phi}\,;\hat{p}) - L(\hat{\phi}\,;\hat{p}) \end{aligned} \tag{A16}$$

From $\delta\phi(\hat{x})$ we can define $\delta\phi(x)$ and $\delta\phi(p)$ [cf. (A11) and (A12)], which are assumed to vanish at four-dimensional infinities in $x$ and $p$ spaces, respectively.

A simple but relevant example is

$$\hat{L} = -\tfrac{1}{2}[\phi(\hat{x})\hat{p}^2\phi(\hat{x}) + M^2\phi^2(\hat{x})] \tag{A17}$$

Because $\hat{p}^\mu|0\rangle = 0$, we can replace $\hat{\phi}\hat{p}^2\hat{\phi}$ with $[\hat{\phi}, \hat{p}_\mu][\hat{p}^\mu, \hat{\phi}] = (\hat{\partial}_\mu\hat{\phi})(\hat{\partial}^\mu\hat{\phi})$. Hence

$$\mathscr{L}(x) = -\tfrac{1}{2}\{[\partial_\mu\phi(x)]^2 + M^2\phi^2(x)\} \tag{A18}$$

Similarly

$$\mathscr{L}(p) = -\tfrac{1}{2}[\phi(-p)p^2\phi(p) + M^2\phi(-p)\phi(p)]. \tag{A19}$$

The advantage of this formalism is evident when $\hat{L}$ becomes a complicated function of $\hat{p}^\mu$, as in the case of the dispersion of electrodynamics in a dielectric medium (Watson and Jauch, 1949); one can still deal with such a theory straightforwardly if one works in the momentum space (Šoln, 1981, 1982).

In the text on various occasions we have used the generalized definition of a "dot product" for contracting indices between vectors and second-rank tensors. This generalized definition is explained here with $a^\mu$, $b^\mu$, and $T^{\mu\nu}$:

$$\begin{gathered} a\cdot b = a_\mu b^\mu, \qquad (a\cdot T)^\mu = a_\nu T^{\nu\mu}, \qquad (T\cdot b)^\mu = T^{\mu\nu}b_\nu \\ a\cdot T\cdot b = a_\mu T^{\mu\nu}b_\nu, \qquad b\cdot(a\cdot T) = a\cdot T\cdot b \\ (T\cdot b)\cdot a = a\cdot T\cdot b \end{gathered} \tag{A20}$$

When we transform position and momentum operators $\hat{x}^\mu$ and $\hat{p}^\mu$ into $\hat{y}^\mu$ and $\hat{P}^\mu$, respectively, commutator (A6) must remain invariant. Hence we can write

$$\hat{y}^\mu = t^\mu_\nu \hat{x}^\nu, \qquad \hat{p}^\mu = t^\mu_\nu \hat{P}^\nu \tag{A21}$$

where $t^\mu_\nu$ is some transformation matrix. Since also $y^\mu = (t\cdot x)^\mu$ and $p^\mu = (t\cdot P)^\mu$, then from completeness relations

$$\int d^4y|y\rangle\langle y| = \int d^4x|x\rangle\langle x| = 1$$

and

$$\int d^4P|P\rangle\langle P| = \int d^4p|p\rangle\langle p| = 1$$

we conclude

$$|x\rangle = t^{1/2}|y\rangle, \qquad |P\rangle = t^{1/2}|p\rangle \tag{A22}$$

where $t = \det(t^{\mu}_{\nu})$.

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